

TOTALLY GEODESIC HYPERSURFACES OF HOMOGENEOUS SPACES

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ABSTRACT. We show that a simply connected Riemannian homogeneous space M which admits a totally geodesic hypersurface F is isometric to either (a) the Riemannian product of a space of constant curvature and a homogeneous space, or (b) the warped product of the Euclidean space and a homogeneous space, or (c) the twisted product of the line and a homogeneous space (with the warping/twisting function given explicitly). In the first case, F is also a Riemannian product; in the last two cases, it is a leaf of a totally geodesic homogeneous fibration. Case (c) can alternatively be characterised by the fact that M admits a Riemannian submersion onto the universal cover of the group $SL(2)$ equipped with a particular left-invariant metric, and F is the preimage of the two-dimensional solvable totally geodesic subgroup.

1. INTRODUCTION

The study of totally geodesic submanifolds of homogeneous spaces dates back to the classical result of Élie Cartan from 1927 ([C] or [Hel, IV, §7]), which says that a totally geodesic submanifold of a symmetric space is the exponent of a Lie triple system. Homogeneous totally geodesic submanifolds of nilpotent Lie groups have been extensively studied in [Ebe, KP, CHN1, CHN2]. The classification of totally geodesic submanifolds of nonsingular two-step nilpotent Lie groups is given in [Ebe].

In the last two decades, a remarkable progress has been achieved in the study of one-dimensional totally geodesic submanifolds — *homogeneous geodesics* (the geodesics which are the orbits of a one-dimensional isometry group); this includes the deep existence results [Kai, KS, Dus] and the investigation of the *g.o. spaces* — homogeneous spaces all of whose geodesics are homogeneous (see e.g. [Gor, AN]).

In this paper we investigate the other extremity — totally geodesic hypersurfaces (not necessarily homogeneous) of homogeneous spaces. As one may expect, the existence of such a hypersurface imposes strong restrictions on the ambient space. In particular, if a homogeneous space admits a totally geodesic hypersurface, then it must be a space of constant curvature, provided it belongs to one of the following classes: irreducible symmetric spaces [CN], normal homogeneous spaces [To2], and more generally, naturally reductive homogeneous spaces [Ts1, To1]. Totally geodesic hypersurfaces and extrinsic hyperspheres in manifolds with special holonomy have been recently studied in [JMS]. By [CHN1, Proposition 5], if a nilmanifold admits a totally geodesic homogeneous hypersurface F , then its metric Lie algebra is the direct orthogonal sum of a one-dimensional ideal and the ideal tangent to F .

We prove the following classification theorem.

Theorem 1. *Suppose M is a simply connected, connected Riemannian homogeneous space and $F \subset M$ is a complete connected totally geodesic hypersurface. Then one of the following holds.*

- (a) $M = M_1(c) \times M_2$, the **Riemannian product** of a space $M_1(c)$ of constant curvature c and a homogeneous space M_2 . The hypersurface F is the product $F_1(c) \times M_2$, where $F_1(c) \subset M_1(c)$ is totally geodesic.
- (b) $M = \mathbb{R}^m_f \times M_2$, the **warped product** of \mathbb{R}^m , $m > 0$, and a homogeneous space $M_2 = G/H$, with the warping function $f : M_2 \rightarrow \mathbb{R}$ defined by $f(gH) = \chi(g)$, where $\chi : G \rightarrow (\mathbb{R}^+, \cdot)$ is a nontrivial homomorphism with $\chi(H) = 1$. The hypersurface F is the Cartesian product of a hyperplane $\mathbb{R}^{m-1} \subset \mathbb{R}^m$ and M_2 .
- (c) $M = \mathbb{R}_f \times M_2$, the **twisted product** of \mathbb{R} and a homogeneous space M_2 . The hypersurface F is a leaf of the totally geodesic fibration $\{t\} \times M_2$, $t \in \mathbb{R}$.

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Moreover, the curves $\mathbb{R} \times \{x\}$, $x \in M_2$, are congruent helices of order two with the curvature k and the torsion $\kappa \neq 0$. With a particular choice of local coordinates t on \mathbb{R} and u on M_2 , the twisting function is given by $f(t, u) = (\sinh(\alpha(u)) \cos(\kappa t + \beta(u)) + \cosh(\alpha(u)))^{-2}$, where locally $\alpha, \beta : M_2 \rightarrow \mathbb{R}$ satisfy $\|\nabla \alpha\|^2 = \sinh(\alpha)^2 \|\nabla \beta\|^2 = k^2$.

The warped (the twisted) product $M_1 \times_f M_2$ of Riemannian manifolds (M_1, ds_1^2) and (M_2, ds_2^2) , with the warping function $f : M_2 \rightarrow \mathbb{R}^+$ (respectively, with the twisting function $f : M_1 \times M_2 \rightarrow \mathbb{R}^+$), is the Cartesian product $M_1 \times M_2$ equipped with the metric $f ds_1^2 + ds_2^2$. A smooth curve in a Riemannian space is called a *helix of order* $p \geq 0$, if its first p Frenet curvatures are nonzero constants and the $(p+1)$ -st Frenet curvature vanishes (by analogy with curves in \mathbb{R}^3 , for helices of order two, we call the first two nonzero curvatures the *curvature* and the *torsion*, respectively). Note that we impose the assumption of completeness of F only for convenience; any open portion of a totally geodesic hypersurface of M can be extended to a complete hypersurface by extending all the geodesics.

It follows from Theorem 1 that apart from Case (a), a totally geodesic hypersurface F is a leaf of a totally geodesic fibration of codimension one.

Theorem 1 is intentionally stated in a purely ‘‘Riemannian’’ language (except for a small amount of algebra in Case (b)) avoiding the choice of a particular presentation of M as G/H . An important question in the theory of totally geodesic submanifolds of homogeneous spaces is when such a submanifold is *homogeneous* (that is, is the orbit of a subgroup of G). From Theorem 2 below (or from the proof of Theorem 1 given in Section 2) one can deduce that in Case (c) of Theorem 1, the hypersurface F is homogeneous relative to any choice of a connected transitive group G of isometries of M . The answer in the other two cases depends on a particular presentation. In Case (a) it can easily be in negative: the group $SU(2)$ with a metric of constant positive curvature contains no two-dimensional subgroups. An example of a non-homogeneous totally geodesic hypersurface from Case (b) is given in Section 2. Note however that in all the cases, the Riemannian manifold F is homogeneous relative to the induced metric.

Theorem 1 has the following obvious but useful corollary.

Corollary. *A compact, simply connected, connected Riemannian homogeneous space that admits a totally geodesic hypersurface F is the Riemannian product of a standard sphere S^m , $m \geq 2$, and a compact homogeneous space M_2 ; then F is (a domain of) the product of a great hypersphere S^{m-1} and M_2 .*

One can give an alternative, more algebraic description of the totally geodesic hypersurface from Case (c) of Theorem 1. The ‘‘smallest’’ example of such a hypersurface is constructed as follows.

Example 1. Take $M = \widetilde{SL(2)}$, the universal cover of the group $SL(2)$. Denote $\mathfrak{g} = \mathfrak{sl}(2)$. Let $\mathfrak{f} \subset \mathfrak{g}$ be a two-dimensional subalgebra and let $N \in \mathfrak{g}$ span the one-dimensional subalgebra $\mathfrak{so}(2) \subset \mathfrak{g}$. Up to automorphism and scaling one can choose, in the defining representation of \mathfrak{g} ,

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathfrak{f} = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Introduce an inner product on \mathfrak{g} by requiring that $N \perp \mathfrak{f}$ and by specifying it further on \mathfrak{f} in such a way that the operator $\pi_{\mathfrak{f}} \text{ad}_N \pi_{\mathfrak{f}}$ is skew-symmetric. Explicitly, choose arbitrary nonzero $a, b \in \mathbb{R}$ and define the inner product $\langle \cdot, \cdot \rangle$ in such a way that the following basis is orthonormal:

$$(1) \quad E_1 = aN = a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = 2b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then \mathfrak{f} is a totally geodesic subalgebra of the metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ [Ts2, Theorem 7.2], and so the subgroup F_1 tangent to \mathfrak{f} is a totally geodesic hypersurface of $M = \widetilde{SL(2)}$ equipped with the left-invariant metric obtained from the inner product (1). Note that F_1 is isometric to the hyperbolic space and the functions α, β from Case (c) are, up to scaling, the polar coordinates on F_1 .

Theorem 2. *Under the assumptions of Theorem 1, either the pair $(M = G/H, F)$ belongs to one of the cases (a), (b), or otherwise there exists a normal subgroup $N \subset G$ such that $H \subset N$, $G/N \simeq \widetilde{SL(2)}$, and the projection $\pi : M \rightarrow \widetilde{SL(2)}$ (where the metric on $\widetilde{SL(2)}$ is constructed as in Example 1) is a Riemannian submersion, and $F = \pi^{-1}F_1$.*

2. PROOFS

Let $M = G/H$ be a simply connected, connected Riemannian homogeneous space, with G a simply connected, closed, connected transitive group of isometries acting on M from the left and H the (connected) isotropy subgroup of a point $o \in M$. Let $\pi : G \rightarrow M$ be the natural projection with $\pi(e) = o$. Denote $\langle \cdot, \cdot \rangle$, ∇ and R the metric, the Levi-Civita connection and the curvature tensor of M respectively. For vector fields $X, Y \in TM$ we define the operator $X \wedge Y \in \mathfrak{so}(TM)$ by $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$. Denote $\mathcal{R} \in \text{Sym}(\mathfrak{so}(TM))$ the curvature operator, the symmetric operator defined by $\langle \mathcal{R}(X \wedge Y), Z \wedge V \rangle = \langle R(X, Y)Z, V \rangle$, where the inner product on the left-hand side is the natural inner product on $\mathfrak{so}(TM)$. For a vector X and a subspace V we denote $X \wedge V$ the subspace $\text{Span}(X \wedge Y : Y \in V)$.

Let $F \in M$ be a connected totally geodesic hypersurface. Without loss of generality we can assume that $o \in F$. Moreover, as M is an analytic Riemannian manifold and as F is totally geodesic, hence minimal, it is an analytic submanifold of M . Therefore we can (and will) replace F by a small open disc of F containing o . Let ξ be a continuous unit vector field normal to F . Consider the *Gauss image* of F defined by $\Gamma(F) = \{dg^{-1}\xi(x) : x \in F, g \in G, g(o) = x\}$. The set of pairs $(x, g) \in F \times G$ such that $g(o) = x$ is (locally) diffeomorphic to $\pi^{-1}F \simeq F \times H$, so $\Gamma(F)$ is the image of a continuous (in fact, analytic) ‘‘Gauss map’’ $\Phi : \pi^{-1}F \rightarrow S_o(1)$, where $S_o(1)$ is the unit sphere of T_oM . As H is connected, $\Gamma(F)$ is also connected. Moreover, $\Gamma(F)$ is H -left-invariant. Then the subspace $D_o = \text{Span}(\Gamma(F)) \subset T_oM$ is H -left-invariant, as also is its orthogonal complement D_o^\perp . Hence we can define two orthogonal complementary G -left-invariant distributions D and D^\perp on M such that $D(o) = D_o$ and $D^\perp(o) = D_o^\perp$. Denote $m = \dim D$.

Lemma 1. *In the above notation we have:*

1. *The distribution D^\perp is integrable with totally geodesic leaves. The leaf of D^\perp passing through o locally lies in F .*
2. *The distribution D is integrable with totally umbilical leaves.*
3. *$D \wedge D$ lies in an eigenspace of \mathcal{R} , so that there exists $\lambda \in \mathbb{R}$ such that $\mathcal{R}(X \wedge Y) = \lambda X \wedge Y$, for all $X, Y \in D$.*

Proof. 1. First note that if X is tangent to D^\perp at some point $x \in F$ and $x = g(o)$, then $dg^{-1}X \in D_o^\perp$ (as D^\perp is G -left-invariant), hence $dg^{-1}X \perp dg^{-1}\xi(x)$, so $X \perp \xi(x)$. Thus D^\perp is tangent to F .

Now let X, Y be two vector fields tangent to D^\perp in a neighbourhood of o . They must be tangent to F at the points of F . As F is totally geodesic, we have $(\nabla_X Y)|_o \perp \xi(o)$. Moreover, for any $x \in F$ and any $g \in G$ such that $g(o) = x$, the vector field dgX and dgY are tangent to D and to F (at the points of F), so $(\nabla_{dgX} dgY)|_x \perp \xi(x)$, hence $(\nabla_X Y)|_o \perp dg^{-1}\xi(x)$. It follows that $(\nabla_X Y)|_o \in D_o^\perp$. As D^\perp is G -left-invariant it follows that everywhere on G we have $\nabla_X Y \in D^\perp$, for any vector fields $X, Y \in D^\perp$. Therefore $[D^\perp, D^\perp] \subset D^\perp$, and the leaves tangent to D^\perp are totally geodesic submanifolds of M .

2. Let $\eta \in \Gamma(F)$ and let $g \in G$ and $x \in F$ be chosen in such a way that $\eta = dg^{-1}\xi(x)$. Let $Z' \in T_x F \cap D(x)$ and let $X' \in D^\perp$ be a vector field in a neighbourhood of x . Then $(\nabla_{Z'} X')|_x \perp \xi(x)$. Acting by dg^{-1} we obtain that $(\nabla_Z X)|_o \perp \eta$ for any $Z \in D_o \cap \eta^\perp$ and for any vector field $X \in D$ in a neighbourhood of o . It follows that every $\eta \in \Gamma(F)$ is an eigenvector of the linear operator L_X on D_o defined by $\langle L_X N_1, N_2 \rangle = \langle (\nabla_{N_2} X)|_o, N_1 \rangle$ (L_X is the adjoint to the Nomizu operator of X). As $\Gamma(F)$ is a connected subset of the unit sphere of T_oM spanning D_o we obtain that L_X is proportional to the identity, so that the bilinear form on $D_o \times D_o$ defined by $(Z_1, Z_2) \mapsto \langle (\nabla_{Z_1} X)|_o, Z_2 \rangle$ vanishes for all $Z_1 \perp Z_2$, $Z_1, Z_2 \in D_o$. Let $N_1, N_2 \in D$ be orthogonal vector fields in a neighbourhood of o (if $m (= \dim D) = 1$, the claim of the assertion is trivial) and let $X \in D^\perp$ be a vector field in a neighbourhood of o . At the point o we have $\langle \nabla_{N_1} N_2, X \rangle = -\langle \nabla_{N_1} X, N_2 \rangle = 0$. It follows that $[N_1, N_2] \in D$ for any two orthogonal vector fields $N_1, N_2 \in D$, hence for any such N_1, N_2 . Then D is integrable and the second fundamental form of the leaves vanishes on any pair of orthogonal vectors. It follows that the second fundamental form (in every direction from D^\perp) is proportional to the induced inner product on D , hence the leaves are totally umbilical.

3. Let $g \in \pi^{-1}F$ with $x = g(o) \in F$. From the Codazzi equation at x we have $\langle R(X, Y)Z, \xi \rangle = 0$, for all $X, Y, Z \in T_x F$. From the symmetries of the curvature tensor it follows that $\langle R(\xi, X)Y, Z \rangle = \langle R_\xi X, Z \rangle \langle \xi, Y \rangle - \langle R_\xi X, Y \rangle \langle \xi, Z \rangle$, for all $X, Y, Z \in T_x M$, where $R_\xi : T_x M \rightarrow T_x M$ is the Jacobi operator defined by $R_\xi X = R(\xi, X)\xi$. Then $\mathcal{R}(\xi \wedge X) = (R_\xi X) \wedge \xi$. As R_ξ is symmetric, there exists an orthonormal basis e_i , $i = 1, \dots, n-1$, for $T_x F$ such that $\mathcal{R}(\xi \wedge e_i) = c_i \xi \wedge e_i$, so that the elements $\xi \wedge e_i \in \mathfrak{so}(T_x M)$ are the eigenvectors of $\mathcal{R} \in \text{Sym}(\mathfrak{so}(T_x M))$ [Ts2, Proposition 4.7]. Acting by dg^{-1} , we obtain that for every $g \in \pi^{-1}F$ there is a direct orthogonal decomposition $T_o M = \mathbb{R}\Phi(g) \oplus \bigoplus_{s=1}^{p(g)} L_s(g)$ (where Φ is the Gauss map and $\Phi(g) \in \Gamma(F)$) such that every subspace $\Phi(g) \wedge L_s(g) \subset \mathfrak{so}(T_o M)$ lies in the eigenspace of $\mathcal{R} \in \text{Sym}(\mathfrak{so}(T_o M))$ with the eigenvalue $\lambda_s(g)$ (here $\lambda_s(g)$'s are the c_i 's without repetitions). Let $\mathfrak{so}(T_o M) = \bigoplus_{a=1}^N V_a$ be the orthogonal decomposition of $\mathfrak{so}(T_o M)$ on the eigenspaces of \mathcal{R} , with μ_a the corresponding eigenvalues. Then every $\lambda_s(g)$ equals to one of the constants μ_a . As the Jacobi operator R_ξ depends continuously (in fact, analytically) on $g \in \pi^{-1}F$ and all its eigenvalues belong to the finite set $\{\mu_a\}$ we obtain that the number of eigenvalues $p(g) = p$ is constant and up to relabelling, every subspace $\Phi(g) \wedge L_s(g)$ lies in V_s . Moreover the dimensions $m_s = \dim L_s(g)$ are constant and the maps $g \mapsto L_s(g)$ are analytic maps from $\pi^{-1}F$ to the Grassmanians $G(m_s, T_o M)$. It follows that for any $g, h \in \pi^{-1}F$ and for any $s \neq l$, we have $\langle \Phi(g) \wedge L_s(g), \Phi(h) \wedge L_l(h) \rangle = 0$, so $(\Phi(g) \wedge L_s(g)) \Phi(h) \perp L_l(h)$, therefore $(\Phi(g) \wedge L_s(g)) \Phi(h) \subset L_s(h)$. Now if $\Phi(h) \not\subset \Phi(g)$, the subspace $(\Phi(g) \wedge L_s(g)) \Phi(h)$ has dimension m_s , the same as the dimension of $L_s(h)$. So there exists a small enough neighbourhood $\mathcal{U} \subset \pi^{-1}F$ of e such that for all $g, h \in \mathcal{U}$ and all $s = 1, \dots, p$, we have $(\Phi(g) \wedge L_s(g)) \Phi(h) = L_s(h)$, hence $L_s(h) \subset \mathbb{R}\Phi(g) \oplus L_s(g)$. Let $N_s = \text{Span}(L_s(h) : h \in \mathcal{U})$. Then $\dim N_s \geq m_s$ as $\dim L_s(h) = m_s$, and moreover, since $N_s \subset \mathbb{R}\Phi(g) \oplus L_s(g)$, for all $g \in \mathcal{U}$, we have $\dim N_s \leq m_s + 1$. So we have two possibilities: either $\dim N_s = m_s$, in which case the subspaces $L_s(h)$ do not depend on h : $L_s(h) = N_s$, for all $h \in \mathcal{U}$; or $\dim N_s = m_s + 1$, in which case the subspaces $\mathbb{R}\Phi(g) \oplus L_s(g)$ do not depend on g : $\mathbb{R}\Phi(g) \oplus L_s(g) = N_s$, for all $g \in \mathcal{U}$. But the latter case occurs for no more than one $s = 1, \dots, p$. Indeed, if we suppose that $\mathbb{R}\Phi(g) \oplus L_s(g) = N_s$ and $\mathbb{R}\Phi(g) \oplus L_l(g) = N_l$, for all $g \in \mathcal{U}$ and for some $s \neq l$, then, as $\Phi(g), L_s(g)$ and $L_l(g)$ are mutually orthogonal, we obtain $\mathbb{R}\Phi(g) = N_s \cap N_l$. It follows that $\Phi(g)$ is constant, and so the subspaces $L_s(g) = N_s \cap (\Phi(g))^\perp$ also do not depend on g . But then $L_s(g) = \text{Span}(L_s(h) : h \in \mathcal{U}) = N_s$, so $\dim N_s = m_s$, a contradiction. So $\dim N_s = m_s + 1$ for no more than one $s = 1, \dots, p$, and $\dim N_s = m_s$ for all the other s .

Now if $\dim N_s = m_s$ for all $s = 1, \dots, p$, then the vector $\Phi(g)$ and all the subspaces $L_s(g)$ are constant: $\Phi(g) = \Phi$ and $L_s(g) = N_s$, for all $g \in \mathcal{U}$, hence for all $g \in \pi^{-1}F$, by analyticity. Then the distribution D is one dimensional and the claim follows trivially.

Otherwise, suppose that $\dim N_1 = m_1 + 1$. Then again $L_s(g) = N_s$, for all $g \in \pi^{-1}F$ and for all $s \geq 2$. We also have $D_o = \text{Span}(\Phi(g) : g \in \pi^{-1}F) = \text{Span}(\Phi(g) : g \in \mathcal{U})$ by analyticity, so $D_o \subset N_1$. But for any $g \in \pi^{-1}F$, we have $\Phi(g) \wedge E_1(g) = \Phi(g) \wedge N_1 \subset V_1$ and V_1 is the eigenspace of \mathcal{R} with the eigenvalue λ_1 . It follows that $\Phi(g) \wedge D_o \subset V_1$, for all $g \in \pi^{-1}F$, hence $D_o \wedge D_o \subset V_1$, as required. \square

Proof of Theorem 1. Let $\nu \in D^\perp$ be the mean curvature vector field of the totally umbilical foliation on M defined by D . As D is G -left-invariant, ν is also G -left-invariant.

We consider two cases for $m = \dim D$.

Case 1. Suppose that $m (= \dim D) > 1$. Then from Codazzi equation and from Lemma 1(3) we obtain that the D^\perp component of the vector field $\langle Z_1, Z_3 \rangle \nabla_{Z_2} \nu - \langle Z_2, Z_3 \rangle \nabla_{Z_1} \nu$ vanishes, for any Z_1, Z_2 and Z_3 tangent to D . It follows that $\nabla_Z \nu$ is tangent to D , for any Z tangent to D , hence the leaves of the foliation defined by D are *extrinsic spheres*.

We can introduce analytic local coordinates $v^1, \dots, v^m, u^1, \dots, u^{n-m}$ in a neighbourhood of any point $x \in M$ in such a way that $D = \text{Span}(\partial/\partial v^\alpha : \alpha = 1, \dots, m)$, $D^\perp = \text{Span}(\partial/\partial u^i : i = 1, \dots, n-m)$. The metric of M is given by $ds^2 = A'_{\alpha\beta}(u, v) dv^\alpha dv^\beta + B'_{ij}(u, v) du^i du^j$. As D^\perp is totally geodesic, we obtain that $B'_{ij} = B_{ij}(u)$. From the fact that D is totally umbilical we get $A'_{\alpha\beta}(u, v) = f(u, v) A_{\alpha\beta}(v)$ for some positive analytic function f . Then $\nu = -\frac{1}{2} B^{ij} \partial(\ln f) / \partial u^i \partial/\partial v^j$ and the fact that the leaves tangent to D are extrinsic spheres gives $\partial^2(\ln f) / \partial u^i \partial v^\alpha = 0$. It follows that f is a product of a function of the u 's by a function of the v 's, so (with a slight change of notation) $ds^2 = f(u) A_{\alpha\beta}(v) dv^\alpha dv^\beta + B_{ij}(u) du^i du^j$, hence M is locally a warped product. Moreover, from Gauss equation and from Lemma 1(3), every leaf

tangent to D has a constant curvature in the induced metric. But the isometry of M which maps a point $x = (u, v)$ to a point $y = (u', v)$ on the same leaf is a homothety with the coefficient $f(u')/f(u)$. It follows that either f is constant or every leaf tangent to D is flat in the induced metric.

Now, if f is a constant, then M is locally a Riemannian product. As M is simply connected, by de Rham Theorem, it is the Riemannian product of a leaf M_1 tangent to D and a leaf M_2 tangent to D^\perp , with both M_1 and M_2 homogeneous (as D and D^\perp are G -left-invariant). Moreover, by Lemma 1(3), M_1 has a constant curvature c . Let F_1 be (the unique complete) totally geodesic hypersurface of $M_1(c)$ whose normal vector at o is $\xi(o)$. Then the hypersurface $F' = F_1 \times M_2$ is totally geodesic and F is an open subset of F' (as a totally geodesic submanifold is locally uniquely determined by its tangent space at a single point). This gives **Case (a) of Theorem 1**.

Now suppose that f is not a constant. From the above and by [BH, Theorem A] the manifold M is a global warped product, $M = \mathbb{R}^m_f \times M_2$, where M_2 is the leaf tangent to D^\perp passing through o and $f : M_2 \rightarrow \mathbb{R}^+$. The isotropy subgroup $G_2 \subset G$ of M_2 acts transitively and isometrically on M_2 , so M_2 is the homogeneous space G_2/H . Moreover, every $g \in G_2$ acts on the \mathbb{R}^m fibers by the homothety with the coefficient $f(g(u))/f(u)$. As this ratio must not depend of $u \in M_2$ we obtain that $f(gH) = \chi(g)$, where $\chi : G_2 \rightarrow (\mathbb{R}^+, \cdot)$ is a homomorphism with $\chi(H) = 1$. Let \mathbb{R}^{m-1} be the hyperplane of \mathbb{R}^m passing through o and orthogonal to $\xi(o)$. Then the hypersurface $F \subset M$, the (Cartesian) product of \mathbb{R}^{m-1} and M_2 is totally geodesic and is (the unique complete) totally geodesic hypersurface of M whose normal vector at o is $\xi(o)$. This gives **Case (b) of Theorem 1**.

Case 2. Suppose that $m (= \dim D) = 1$. Let τ be a unit vector field on M which spans D (so that $\tau(o) = \xi(o)$). By construction, τ is G -left-invariant. Moreover, from [BH, Theorem A] the manifold M is diffeomorphic to $\mathbb{R} \times M_2$, where M_2 is the leaf of D^\perp passing through o . The leaf M_2 is a totally geodesic hypersurface and F is an open connected subset of M_2 . Let $G_1 \subset G$ be the connected isotropy subgroup of M_2 . Then $H \subset G_1$ and G_1 has codimension one in G . It follows that F is a homogeneous totally geodesic hypersurface.

Now, if the vector field τ is geodesic, then we get back to case Case (a), with $M_1(c)$ a Euclidean line. Furthermore, if τ is not geodesic, but the leaves of D are “circles” (one-dimensional extrinsic spheres), that is, if $\nabla_\tau \nu = -\|\nu\|^2 \tau$, then repeating the above arguments we get to Case (b), with $m = 1$.

Suppose that $\nabla_\tau \nu \not\parallel \tau$. Consider the Frenet frame $\tau, \nu_1, \nu_2, \dots$ of the one-dimensional leaves of D . We have $\nabla_\tau \tau = k_1 \nu_1 (= \nu)$, $\nabla_\tau \nu_1 = -k_1 \tau + k_2 \nu_2$. By the G -left-invariance, all the Frenet curvatures k_1, k_2, \dots are constant; by our assumption, at least the first two of them, k_1 and k_2 , are nonzero.

Passing to the level of Lie algebras, we need to exercise a certain caution, as the standard identification procedure is carried out via Killing vector fields, however the vector fields $\tau, \nu_1, \nu_2, \dots$ are not in general Killing. Denote their values at o by the corresponding Roman letters, so that $T = \tau(o)$, $N_1 = \nu_1(o)$, $N_2 = \nu_2(o)$, etc. Note that the spans of the vectors T, N_1, N_2, \dots are one-dimensional H -submodules of $T_o M$. Let $\mathfrak{g}, \mathfrak{g}_1$ and \mathfrak{h} be the Lie algebras of G, G_1 and H respectively. We have $\mathfrak{h} \subset \mathfrak{g}_1 \subset \mathfrak{g}$, with \mathfrak{g}_1 a subalgebra of codimension one in \mathfrak{g} . Choose and fix an $\text{ad}(H)$ -invariant complement \mathfrak{f} to \mathfrak{h} in \mathfrak{g}_1 . The corresponding Killing vector fields are tangent to D^\perp , and we can identify \mathfrak{f} with $D^\perp(o)$. As the inner product is $\text{ad}(H)$ -invariant, we can find a one-dimensional $\text{ad}(H)$ -invariant complement to \mathfrak{g}_1 in \mathfrak{g} spanned by an element whose corresponding Killing vector field at o equals T . We can identify that element with T , and the space $\mathfrak{m} = \mathbb{R}T \oplus \mathfrak{f} \subset \mathfrak{g}$, with $T_o M$. Then we obtain

$$(2) \quad \mathfrak{m} = \mathbb{R}T \oplus \mathfrak{f}, \quad T \perp \mathfrak{f}, \quad \mathfrak{g}_1 = \mathfrak{f} \oplus \mathfrak{h}, \quad [T, \mathfrak{h}] = 0, \quad [\mathfrak{h}, \mathfrak{f}] \subset \mathfrak{f}, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1.$$

We have the following lemma:

Lemma 2.

- (a) *The leaves of D are helices of order two: their first and second Frenet curvatures are nonzero constants, and the third Frenet curvature is zero.*
- (b) *Denote $\mathfrak{l} = \text{Span}(T, N_1, N_2)$, $\mathfrak{s} = \text{Span}(N_1, N_2)$ and $\mathfrak{J} = (\mathfrak{m} \cap \mathfrak{l}^\perp) \oplus \mathfrak{h}$. Then*
 - (i) *\mathfrak{J} is an ideal of \mathfrak{g} containing \mathfrak{h} , with $\mathfrak{g}/\mathfrak{J} \simeq \mathfrak{sl}(2)$.*
 - (ii) *$\mathfrak{g}_1 = \mathfrak{s} \oplus \mathfrak{J}$.*
 - (iii) *$[\mathfrak{l}, \mathfrak{h}] = 0$.*

- (c) The subspace $\mathfrak{l} \subset \mathfrak{g}$, with the induced inner product and with the Lie algebra structure of $\mathfrak{g}/\mathfrak{J}$, is isometrically isomorphic to $\mathfrak{sl}(2)$ with the metric (1), with $\mathfrak{s} \subset \mathfrak{l}$ the totally geodesic solvable subalgebra defined by $\mathfrak{s} = \text{Span}(E_2, E_3)$.

Proof. (a and b) From the fact that \mathfrak{f} is tangent to a totally geodesic hypersurface (and that $\mathfrak{g}_1 \subset \mathfrak{g}$ is a subalgebra) we obtain that for all $X, Y \in \mathfrak{f}$,

$$(3) \quad \langle [T, X]_{\mathfrak{m}}, Y \rangle + \langle [T, Y]_{\mathfrak{m}}, X \rangle = 0,$$

where the subscript \mathfrak{m} denotes the \mathfrak{m} -component. To compute the Frenet frame we shall use the following fact: if \tilde{Y} and \tilde{Z} are G -left-invariant vector fields on M with $\tilde{Y}(o) = Y$, $\tilde{Z}(o) = Z$, then

$$(4) \quad (\nabla_{\tilde{Y}} \tilde{Z})(o) = \frac{1}{2}[Y, Z]_{\mathfrak{m}} + U(Y, Z), \quad \text{where } 2\langle U(Y, Z), X \rangle = \langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle [X, Z]_{\mathfrak{m}}, Y \rangle,$$

for all $X \in \mathfrak{m}$. Note that the first term on the right-hand side of (4) differs by the sign from that in the standard formula for the covariant derivative of Killing vector fields (e.g. [Bes, Proposition 7.28]); equation (4) easily follows from that formula and the fact that the Lie bracket of a (G -)Killing vector field and a G -left-invariant vector field (as of vector fields on M) vanishes.

The elements N_1, N_2, \dots of the Frenet frame at o are orthonormal unit vectors in \mathfrak{f} . From (3), (4) and the fact that $k_1\nu_1 = \nabla_{\tau}\tau$ we have

$$(5) \quad \langle T, [X, T]_{\mathfrak{m}} \rangle = k_1 \langle N_1, X \rangle,$$

for all $X \in \mathfrak{m}$. From the Frenet equations we have $\nu_2 = k_2^{-1}k_1\tau + k_2^{-1}\nabla_{\tau}\nu_1$. Then for any $X \in \mathfrak{f}$, we obtain $\langle N_2, X \rangle = \frac{1}{2}k_2^{-1}(\langle [T, N_1]_{\mathfrak{m}}, X \rangle + \langle T, [X, N_1]_{\mathfrak{m}} \rangle + \langle N_1, [X, T]_{\mathfrak{m}} \rangle) = k_2^{-1}\langle [T, N_1]_{\mathfrak{m}}, X \rangle$, since \mathfrak{g}_1 is a subalgebra and by (2), (3), (4). As $\langle \tau, \nu_2 \rangle = 0$ we get from (5)

$$(6) \quad N_2 = k_2^{-1}[T, N_1]_{\mathfrak{m}} + k_2^{-1}k_1T.$$

By a classical result [Lie, T, Hof], a subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$ of codimension one must contain the kernel \mathfrak{i} of a homomorphism from \mathfrak{g} to $\mathfrak{sl}(2)$. Denote \mathfrak{i}' the (linear) projection of \mathfrak{i} to \mathfrak{m} . As $\mathfrak{i} \subset \mathfrak{g}_1$, we have $\mathfrak{i}' \subset \mathfrak{f}$, so $T \perp \mathfrak{i}'$. Moreover, since $\mathfrak{i}' \subset \mathfrak{i} + \mathfrak{h}$ we get by (2) $[T, \mathfrak{i}'] \subset [T, \mathfrak{i}] \subset \mathfrak{i}$, as \mathfrak{i} is an ideal. It follows that $[T, \mathfrak{i}']_{\mathfrak{m}} \subset \mathfrak{i}' \subset \mathfrak{f}$. Taking $X \in \mathfrak{i}'$ in (5) we then get $N_1 \perp \mathfrak{i}'$. Furthermore, taking the inner product of (6) with $X \in \mathfrak{i}'$ we get $\langle N_2, X \rangle = -k_2^{-1}\langle [T, X]_{\mathfrak{m}}, N_1 \rangle$ by (3). But from the above, $[T, \mathfrak{i}']_{\mathfrak{m}} \subset \mathfrak{i}'$ and $N_1 \perp \mathfrak{i}'$, so $N_2 \perp \mathfrak{i}'$. Therefore the codimension of \mathfrak{i} in \mathfrak{g} is at least three (since $\mathfrak{i} \subset \mathfrak{i}' \oplus \mathfrak{h} \subset (\mathfrak{m} \cap \mathfrak{l}^{\perp}) \oplus \mathfrak{h}$), hence it is exactly three, with $\mathfrak{g}/\mathfrak{i} \simeq \mathfrak{sl}(2)$ and $\mathfrak{i}' = \mathfrak{m} \cap \mathfrak{l}^{\perp}$ and $\mathfrak{i} = \mathfrak{i}' \oplus \mathfrak{h}$. In particular, $\mathfrak{h} \subset \mathfrak{i}$. This proves **assertion (b)(i)**, with $\mathfrak{J} = \mathfrak{i}$, and **assertion (b)(ii)** (in view of (2)).

Now from (3), (4) we obtain $\langle (\nabla_{\tau}\nu_2)(o), X \rangle = -\langle [T, X]_{\mathfrak{m}}, N_2 \rangle - \frac{1}{2}\langle [N_2, X]_{\mathfrak{m}}, T \rangle$, for any $X \in \mathfrak{m}$. The right-hand side vanishes for all $X \in \mathfrak{i}'$, and also for $X = N_2$ and $X = T$ (from (5) or from Frenet equations). It follows that $(\nabla_{\tau}\nu_2)(o) = -k_2N_1$, hence $\nabla_{\tau}\nu_2(o) = -k_2\nu_1$, which proves **assertion (a)**.

Note that for any $X, Y \in \mathfrak{m}$ with $[X, \mathfrak{h}] = [Y, \mathfrak{h}] = 0$ we have $[\nabla_X Y, \mathfrak{h}] = 0$. Indeed, for an orthonormal basis e_i for \mathfrak{m} we have $\nabla_X Y = \frac{1}{2}\sum_i (\langle [X, Y]_{\mathfrak{m}}, e_i \rangle + \langle X, [e_i, Y]_{\mathfrak{m}} \rangle + \langle Y, [e_i, X]_{\mathfrak{m}} \rangle) e_i$, so $[Z, \nabla_X Y] = \frac{1}{2}[Z, [X, Y]_{\mathfrak{m}}] + \frac{1}{2}\sum_i (\langle X, [e_i, Y]_{\mathfrak{m}} \rangle + \langle Y, [e_i, X]_{\mathfrak{m}} \rangle) [Z, e_i]$. For the first term on the right-hand side we have: $[Z, [X, Y]_{\mathfrak{m}}] = [Z, [X, Y]]_{\mathfrak{m}} = 0$, where the first equality follows from the fact that both \mathfrak{m} and \mathfrak{h} are $\text{ad}_{\mathfrak{h}}$ -invariant, and the second, from the Jacobi identity. As $[Z, \mathfrak{m}] \subset \mathfrak{m}$ and as ad_Z is skew-symmetric on \mathfrak{m} , we obtain for the second term on the right-hand side (the third one is treated similarly): $\sum_i \langle X, [e_i, Y]_{\mathfrak{m}} \rangle [Z, e_i] = \sum_{i,j} \langle X, [e_i, Y]_{\mathfrak{m}} \rangle \langle [Z, e_i], e_j \rangle e_j = -\sum_{i,j} \langle X, [e_i, Y]_{\mathfrak{m}} \rangle \langle [Z, e_j], e_i \rangle e_j = -\sum_j \langle X, [[Z, e_j], Y]_{\mathfrak{m}} \rangle e_j = \sum_j \langle X, [[e_j, Y], Z]_{\mathfrak{m}} \rangle e_j$, by the Jacobi identity. But $[[e_j, Y], Z]_{\mathfrak{m}} = [[e_j, Y]_{\mathfrak{m}}, Z]$, so $\langle X, [[e_j, Y], Z]_{\mathfrak{m}} \rangle = -\langle [e_j, Y]_{\mathfrak{m}}, [X, Z] \rangle = 0$. So $[\nabla_X Y, \mathfrak{h}] = 0$.

Therefore, as T commutes with \mathfrak{h} , the vectors N_1 and N_2 also do. This proves **assertion (b)(iii)**.

(c) The subspace \mathfrak{l} with the inner product induced from \mathfrak{m} is spanned by the orthonormal vectors T, N_1, N_2 . The fact that $\mathfrak{g}/\mathfrak{i} = \mathfrak{sl}(2)$ follows from the above. Explicitly, for $X, Y \in \mathfrak{l}$ denote $[X, Y]_{\mathfrak{l}}$ the orthogonal projection of $[X, Y]_{\mathfrak{m}}$ to \mathfrak{l} . Then from (6) we get $[T, N_1]_{\mathfrak{l}} = k_2N_2 - k_1T$. Moreover, from (3), $\langle [T, N_2]_{\mathfrak{l}}, N_2 \rangle = 0$ and $\langle [T, N_2]_{\mathfrak{l}}, N_1 \rangle = -\langle [T, N_1]_{\mathfrak{l}}, N_2 \rangle = -k_2$, and from (5) $\langle [T, N_2]_{\mathfrak{l}}, T \rangle = 0$. It follows that $[T, N_2]_{\mathfrak{l}} = -k_2N_1$. Furthermore, as $N_1, N_2 \in \mathfrak{g}_1$, the bracket (in \mathfrak{g}) also lies in \mathfrak{g}_1 , so $[N_1, N_2]_{\mathfrak{l}} \in \text{Span}(N_1, N_2)$. From the Jacobi identity it then follows that $[N_1, N_2]_{\mathfrak{l}} = -k_1N_2$. The

isometric isomorphism between the metric Lie algebras \mathfrak{l} and $\mathfrak{sl}(2)$, with the inner product (1), is given by the correspondence $T = E_1$, $N_1 = -E_3$, $N_2 = E - 2$ and $k_1 = 2b$, $k_2 = 2a$. \square

As the leaves tangent to D are congruent helices of the second order, we will use a more conventional notation for their curvature and torsion: $k = k_1$ and $\kappa = k_2$ respectively.

We can now introduce analytic local coordinates t, u^1, \dots, u^{n-1} in a neighbourhood of any point $x \in M$ in such a way that $D = \text{Span}(\partial/\partial t)$, $D^\perp = \text{Span}(\partial/\partial u^i : i = 1, \dots, n-1)$, $t(x) = u^i(x) = 0$, and the leaf of D passing through x is parametrised by the arclength. Then the metric of M in a neighbourhood of x is given by $ds^2 = e^{2\phi(u,t)} dt^2 + B_{ij}(u,t) du^i du^j$, where ϕ is an analytic function with $\phi(0, t) = 0$ and B is analytic and positively definite. We have

$$(7) \quad \begin{aligned} \tau &= e^{-\phi} \frac{\partial}{\partial t}, & k\nu_1 &= \nabla_\tau \tau = -B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial}{\partial u^j}, & k^2 &= B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j} = \text{const} \neq 0, \\ k\kappa\nu_2 &= \nabla_\tau(k\nu_1) + k^2\tau = -e^{-\phi} B^{ij} \frac{\partial^2 \phi}{\partial u^i \partial t} \frac{\partial}{\partial u^j}, & \nabla_\tau(k\kappa\nu_2) &= -e^{-\phi} B^{ij} \frac{\partial}{\partial t} \left(e^{-\phi} \frac{\partial^2 \phi}{\partial u^i \partial t} \right) \frac{\partial}{\partial u^j}. \end{aligned}$$

As the leaves tangent to D are helices of order two, we have from Frenet equations: $\nabla_\tau(k\kappa\nu_2) = -k\kappa^2\nu_1$, so by (7), $\frac{\partial}{\partial t} \left(e^{-\phi} \frac{\partial^2 \phi}{\partial u^i \partial t} \right) = -\kappa^2 e^\phi \frac{\partial \phi}{\partial u^i}$, for all $i = 1, \dots, n-1$, which gives $\frac{\partial}{\partial u^i} \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \kappa^2 e^{2\phi} \right) = 0$. It follows that $\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \kappa^2 e^{2\phi} = h(t)$, for some analytic function h . But $\phi(0, t) = 0$, so $h(t) = \frac{1}{2} \kappa^2$. This gives the equation $\frac{\partial}{\partial t} \left(e^{-\phi} \left(\frac{\partial \phi}{\partial t} \right)^2 + \kappa^2 (e^\phi + e^{-\phi}) \right) = 0$. Solving this equation we get $e^{-\phi} = \sinh(\alpha(u)) \cos(\kappa t + \beta(u)) + \cosh(\alpha(u))$ for some analytic functions α, β on a neighbourhood of $x \in M_2$. This gives the required expression for the twisting function in **Case (c) of Theorem 1**. A direct calculation using the fact that $k^2 = B^{ij} \frac{\partial \phi}{\partial u^i} \frac{\partial \phi}{\partial u^j}$ from (7) shows that $\|\nabla \alpha\|^2 = \sinh(\alpha)^2 \|\nabla \beta\|^2 = k^2$, as required. \square

As we can see from the proof, in Case (c), the subgroup $G_1 \subset G$ acts transitively on F , so F is a homogeneous totally geodesic hypersurface. To some surprise, there exist non-homogeneous totally geodesic hypersurfaces belonging to Case (b), as the following example shows.

Example 2. Consider the metric solvable Lie algebra with an orthonormal basis Z, X_1, X_2, Y and with the nonzero brackets $[Z, X_1] = X_1 + X_2$, $[Z, X_2] = -X_1 + X_2$, $[Z, Y] = 2Y$. The corresponding left-invariant metric on \mathbb{R}^4 is given by $ds^2 = dz^2 + e^{4z} dy^2 + e^{2z} (dx_1^2 + dx_2^2)$, with $Z = -\frac{\partial}{\partial z}$, $Y = e^{-2z} \frac{\partial}{\partial y}$, $X_1 = e^{-z} \cos z \frac{\partial}{\partial x_1} - e^{-z} \sin z \frac{\partial}{\partial x_2}$, $X_2 = e^{-z} \sin z \frac{\partial}{\partial x_1} + e^{-z} \cos z \frac{\partial}{\partial x_2}$. The resulting homogeneous space indeed belongs to Case (b), with $M_2 = \{x_1 = x_2 = 0\}$, the hyperbolic space. Moreover, the hypersurface $F = \{x_1 = 0\}$ is totally geodesic, but $\text{Span}(Z, Y, X_2)$ is not a subalgebra.

Proof of Theorem 2. Suppose the pair $(M = G/H, F)$ belongs to Case (c) of Theorem 1. In the notation of Lemma 2, let N be the connected (normal) subgroup of G tangent to the ideal \mathfrak{J} . Then by Lemma 2 (b)(i) $N \supset H$ and $G/N = \widetilde{\text{SL}(2)}$ (as the Lie group). Moreover, from Lemma 2 (c) it follows that the projection $\pi : M \rightarrow \widetilde{\text{SL}(2)}$ (defined by $\pi(gH) = gN$ for $g \in G$), where $\widetilde{\text{SL}(2)}$ is equipped with the left-invariant metric defined by the inner product (1) (with some specific choice of the constants a and b), is a Riemannian submersion. Then the projection of F to $\widetilde{\text{SL}(2)}$ is the connected Lie subgroup G_1/N whose Lie algebra is spanned by \mathfrak{s} . \square

From Theorem 2 (or from Lemma 2) it follows that Case (c) of Theorem 1 may only occur if the semisimple part of the Levi-Mal'cev decomposition of the Lie algebra of (any) transitive group of isometries of M contains an ideal isomorphic to $\mathfrak{sl}(2)$.

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